

Generalized Gaussian Quadrature Formulas

BORISLAV D. BOJANOV

*Department of Mathematics, University of Sofia,
Sofia, Bulgaria*

DIETRICH BRAESS

*Institute for Mathematics, Ruhr University at Bochum,
Bochum, West Germany*

AND

NIRA DYN

*School of Mathematical Sciences, Tel-Aviv University,
Tel-Aviv, Israel*

Communicated by Allan Pinkus

Received August 13, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

Existence and uniqueness of canonical points for best L_1 -approximation from an Extended Tchebycheff (ET) system, by Hermite interpolating "polynomials" with free nodes of preassigned multiplicities, are proved. The canonical points are shown to coincide with the nodes of a "generalized Gaussian quadrature formula" of the form

$$\int_a^b u(t) \sigma(t) \operatorname{sign} \prod_{i=1}^n (t-x_i)^{v_i} dt \approx \sum_{i=1}^n \sum_{j=0}^{v_i-2} a_{ij} u^{(j)}(x_i) + \sum_{j=0}^{v_0-1} a_{0j} u^{(j)}(a) + \sum_{j=0}^{v_{n+1}-1} a_{n+1,j} u^{(j)}(b), \quad (*)$$

which is exact for the ET-system. In (*), $\sum_{j=0}^{v_i-2} \equiv 0$ if $v_i = 1$, the $v_i (> 0)$, $i = 1, \dots, n$, are the multiplicities of the free nodes and $v_0 \geq 0$, $v_{n+1} \geq 0$ of the boundary points in the L_1 -approximation problem, $\sum_{i=0}^{n+1} v_i$ is the dimension of the ET-system, and σ is the weight in the L_1 -norm.

The results generalize results on multiple node Gaussian quadrature formulas (v_1, \dots, v_n all even in (*)) and their relation to best one-sided L_1 -approximation. They also generalize results on the orthogonal signature of a Tchebycheff system ($v_0 = v_{n+1} = 0$, $v_i = 1$, $i = 1, \dots, n$, in (*)), and its role in best L_1 -approximation. Recent works of the authors were the first to treat Gaussian quadrature formulas and orthogonal signatures in a unified way. © 1986 Academic Press, Inc.

1. INTRODUCTION

In this paper we study the existence and uniqueness of canonical points for best L_1 -approximation by "polynomials" from an extended Tchebycheff (ET) system, which interpolate the approximated function at free nodes with preassigned multiplicities.

For arbitrary preassigned multiplicities $\mathbf{v} = (v_0, \dots, v_{n+1})$ with $v_i > 0$, $i = 1, \dots, n$, $v_i \geq 0$, $i = 0, n+1$, the canonical points $a = x_0^* < x_1^* < \dots < x_n^* < x_{n+1}^* = b$ determine a "generalize Gaussian quadrature formula" (GGQF) of the form

$$\int_a^b u(t) \sigma(\mathbf{x}^*, \mathbf{v}; t) dt \approx \sum_{i=1}^n \sum_{j=0}^{v_i-2} a_{ij} u^{(j)}(x_i^*) + \sum_{i=0, n+1} \sum_{j=0}^{v_i-1} a_{ij} u^{(j)}(x_i^*), \quad (1.1)$$

which is exact for all functions in the ET-system $U = \text{span}\{u_0, \dots, u_N\}$ of order $N+1 = \sum_{j=0}^{n+1} v_j$. In (1.1) we use the convention $\sum_{j=0}^s \equiv 0$ if $s < 0$, and

$$\sigma(\mathbf{x}, \mathbf{v}; t) = \sigma(t) \text{sign} \prod_{i=1}^n (t - x_i)^{v_i}, \quad (1.2)$$

where $\sigma(t) \in C[a, b]$ is the positive weight function in the L_1 -norm.

The GGQF (1.1) is a generalization, on the one hand, of the multiple node Gaussian quadrature formulas ((1.1) with v_1, \dots, v_n all even), which determine the canonical points for one-sided L_1 -approximation, while on the other hand it generalizes the notion of the orthogonal signature ($v_1 = \dots = v_n = 1$, $v_0 = v_{n+1} = 0$, in (1.1)), which defines the canonical points for best L_1 -approximation from U .

The famous quadrature formulas of Gauss have been an attractive subject of investigation. The classical result of Gauss guarantees the existence of a unique set of n points in the interval of integration such that the corresponding interpolatory quadrature formula is exact for all algebraic polynomials of degree $\leq 2n - 1$. This result was extended during 1950–1955 to arbitrary Tchebycheff systems by Krein [14], and to quadrature formulas with nodes of odd multiplicities in the algebraic case, by Turan [18], Tschakaloff [17], and Popoviciu [16]. The uniqueness of the multiple node formulas in the algebraic case was proved much later by Ghizzetti and Ossicini [8]. The existence, uniqueness, and relation to best L_1 -approximation of the multiple node Gaussian quadrature formula in the case of complete ET-systems were proved by Karlin and Pinkus [12, 13]. Later Barrow [2] proposed a simpler ingenious proof of the existence and

uniqueness based on topological degree, which obviates the requirement of the completeness of the ET-system.

Parallel to this line of investigation the existence of canonical points for best L_1 -approximation, which goes back to Markov, was proved by Hobby and Rice [7]. A simpler proof was given by Pinkus [15], based on the Borsuk Antipodality Theorem.

The unified approach to these two lines of investigation, leading to the notion of GGQF and canonical points for best L_1 -approximation for arbitrary preassigned multiplicities of the nodes of interpolation, was originated by the first author of this paper in [3, 4] and independently by the last two authors in [5]. In the papers [3, 4], the existence, uniqueness, and relation to best L_1 -approximation of the GGQF is proved for the algebraic case, while in [5] these results are conjectured for ET-systems and then used in the proof of the uniqueness of monosplines and perfect splines of least L_1 -norm in a unified framework, via the notion of "generalized monosplines" (see also [6]).

In Section 2 we establish our first main result:

EXISTENCE AND UNIQUENESS THEOREM. *Let $U = \text{span}\{u_0, \dots, u_N\} \subset C^N[a, b]$ be an ET-system and $\sigma \in C[a, b]$ a positive weight function. Then for $v_0 \geq 0, v_{n+1} \geq 0, v_i > 0, i = 1, \dots, n$, such that $N + 1 = \sum_{i=0}^{n+1} v_i$, there exists a unique set of nodes \mathbf{x}^* in*

$$\Omega_n = \{\mathbf{x} = \{x_0, x_1, \dots, x_{n+1}\} : a = x_0 < x_1 < \dots < x_n < x_{n+1} = b\}, \quad (1.3)$$

for which the generalized quadrature formula (GQF)

$$\int_a^b u(t) \sigma(\mathbf{x}^*, \mathbf{v}; t) dt = \sum_{i=0}^{n+1} \sum_{j=0}^{v_i-1} a_{ij} u^{(j)}(x_i^*), \quad u \in U, \quad (1.4)$$

has the property

$$a_{i, v_i-1} = 0, \quad i = 1, \dots, n. \quad (1.5)$$

The proof of this result can be based on arguments from topological degree theory, as a direct extension of the method of Barrow [2]. Here we choose to prove the existence by using the Borsuk Antipodal Theorem as in Pinkus' proof of the Hobby–Rice theorem [15]. The uniqueness is proved by a new inductive argument. In order to use induction, we prove the existence of GQFs with two types of nodes: fixed nodes and Gaussian nodes (see also [9]).

The existence of canonical points for best L_1 -approximation by interpolating "polynomials" with preassigned multiplicities of nodes \mathbf{v} , is proved

in Section 3. More precisely it is shown that for any u_{N+1} such that $\{u_0, \dots, u_{N+1}\}$ is an ET-system of order $N + 2$, the extremal problem

$$\min_{\mathbf{x} \in \Omega_n} \left\{ \int_a^b |u_{N+1}(t) - u(t)| \sigma(t) dt : u \in U, u^{(j)}(x_i) = u_{N+1}^{(j)}(x_i), \right. \\ \left. j = 0, \dots, v_i - 1; i = 0, \dots, n + 1 \right\} \quad (1.6)$$

has a unique solution in Ω_n , independent of u_{N+1} . This solution $\mathbf{x}^* \in \Omega_n$ coincides with the set of nodes of the GGQF (1.1), and the Lagrangian multipliers for the restricted optimization problem (1.6), constitute the coefficients $\{a_{ij}\}$ in (1.1).

Before concluding the Introduction we define, for the sake of completeness, the notion of an ET-system, which is central to this work.

DEFINITION 1. $\{u_0, \dots, u_N\} \subset C^N[a, b]$ is an ET-system on $[a, b]$, if for any $a \leq t_0 \leq \dots \leq t_N \leq b$,

$$\det \{u_i^{(l_j)}(t_j)\}_{i,j=0}^N > 0, \quad (1.7)$$

where $l_j = \max \{m : t_{j-m} = t_j, m \leq j\}, j = 0, \dots, N$.

It follows directly from Definition 1 that any $u \in \text{span} \{u_0, \dots, u_N\} \setminus \{0\}$ has at most N zeros, counting multiplicities, in $[a, b]$, up to order N .

2. EXISTENCE AND UNIQUENESS

The existence of generalized Gaussian quadrature formulas can be easily shown via the Borsuk Antipodal Theorem. Since the latter is usually shown by induction [1], it is probably more than a coincidence that we establish uniqueness by an inductive argument. For this purpose we consider generalized quadrature formulas (GQFs) with two types of nodes: free nodes and fixed (or prescribed) nodes (see also [9]).

THEOREM 1. *Let the conditions of the existence and uniqueness theorem prevail, and let $0 \leq k \leq n$. Given $n - k$ distinct nodes $t_{k+1}, t_{k+2}, \dots, t_n \in (a, b)$, there is a set of k nodes $a = t_0 < t_1 < t_2 < \dots < t_k < t_{n+1} = b$ such that for $\mathbf{t} = (t_0, \dots, t_k, t_{k+1}, \dots, t_{n+1})$*

$$\int_a^b u(t) \sigma(\mathbf{t}, \mathbf{v}; t) dt - \sum_{i=0}^{n+1} \sum_{j=0}^{v_i-1} a_{ij} u^{(j)}(t_i) = 0, \quad u \in U, \quad (2.1)$$

with

$$a_{i,v_i-1} = 0, \quad i = 1, 2, \dots, k. \tag{2.2}$$

Remark 1. The nodes t_1, t_2, \dots, t_k are called the Gaussian nodes of the GQF. Some of them may coincide with some of the prescribed nodes (cf. [9]). In such cases (2.1) must be adjusted as follows: if $t_j = t_l, j \leq k < l$, then t_l has to be removed from the list of fixed nodes and the multiplicity v_j is to be replaced by $v_j + v_l$. Note that such merging of nodes changes neither N nor the sign of $\sigma(\mathbf{t}, \mathbf{v}; t)$.

Our first aim is to understand the fact that the sign changes of $\pm \sigma(\mathbf{t}, \mathbf{v}; t)$ at the Gaussian nodes provide the only sign pattern which is consistent with (2.2) for the multiplicities v_1, \dots, v_k .

LEMMA 1. *Let $1 \leq k \leq n$ and $t_0 < t_1 < \dots < t_k < t_{n+1}$. Assume that $|s(t)| \equiv 1$ and that s is constant in the subintervals $(t_0, t_1), (t_1, t_2), \dots, (t_k, t_{n+1})$. Moreover let $s(t) = +1, t \in [t_0, t_1]$ and \mathbf{t} be as in Theorem 1. If in a formula of the form*

$$\int_a^b u(t) \sigma(\mathbf{t}, \mathbf{v}; t) s(t) dt = \sum_{i=0}^{n+1} \sum_{j=0}^{v_i-1} a_{ij} u^{(j)}(t_i), \quad u \in U, \tag{2.3}$$

(2.2) holds, then $s(t) = +1$ in $(a, b) - \{t_1, \dots, t_k\}$.

Proof. Define an auxiliary multiplicity vector ω by

$$\begin{aligned} \omega_i &= v_i - 1, & \text{if } s \text{ changes its sign at } t_i, 1 \leq i \leq k, \\ &= v_i & \text{otherwise.} \end{aligned}$$

Moreover choose ω_{n+1} such that $\sum_{i=0}^{n+1} (\omega_i - v_i) = 0$. Obviously $\sigma(\mathbf{t}, \omega; t) = s(t) \sigma(\mathbf{t}, \mathbf{v}; t)$ almost everywhere. Consider the function $u \in U$ which solves the interpolation problem

$$u^{(j)}(t_i) = \delta_{i,n+1} \delta_{j,\omega_{n+1}-1}, \quad j = 0, 1, \dots, \omega_i - 1, i = 0, 1, \dots, n + 1. \tag{2.4}$$

From (2.4) it follows that u has N zeros counting multiplicities. Since $\{u_0, u_1, \dots, u_N\}$ is an ET-system, u has no more zeros than specified. Hence $u(t) \sigma(\mathbf{t}, \omega; t)$ is of constant sign in $[a, b]$ and $\int_a^b u \cdot \sigma(\mathbf{t}, \omega; \cdot) \neq 0$. On the other hand $\omega \neq \mathbf{v}$ implies $\omega_{n+1} > v_{n+1}$, and by (2.4) the right-hand side of (2.3) equals zero. This contradiction leads to the conclusion that $s(t)$ has the same sign in $(a, b) - \{t_1, \dots, t_k\}$. ■

The next step towards the proof of Theorem 1 deals with the sign pattern of the leading coefficients in (2.1), corresponding to the Gaussian and the boundary nodes.

LEMMA 2. *If $a_{l, v_l - 1} = 0$ holds for some $l \in \{1, \dots, n\}$ in a GQF of the type (2.1), and if $v_l \geq 2$, then*

$$\text{sign } a_{l, v_l - 2} = \sigma_l := \text{sign } \sigma(\mathbf{t}, \mathbf{v}; t_l + 0). \tag{2.5}$$

Moreover if $v_0 > 0$ in (2.1) then

$$\text{sign } a_{0, j} = \text{sign } \sigma(\mathbf{t}, \mathbf{v}; a + 0), \quad j = 0, \dots, v_0 - 1, \tag{2.6}$$

while if $v_{n+1} > 0$

$$\text{sign } a_{n+1, j} = (-1)^j \sigma(\mathbf{t}, \mathbf{v}; b - 0), \quad j = 0, \dots, v_{n+1} - 1. \tag{2.7}$$

Proof. Choose $u \in U$ according to the N interpolation conditions

$$\begin{aligned} u^{(j)}(t_l) &= \delta_{j, v_l - 2}, & j &= 0, \dots, v_l - 2, \quad u^{(v_{n+1})}(t_{n+1}) = 0, \\ u^{(j)}(t_i) &= 0, & j &= 0, \dots, v_i - 1, \quad i = 0, \dots, n + 1, \quad i \neq l. \end{aligned} \tag{2.8}$$

As in Lemma 1, $u \in U$ has no more zeros than the N specified by (2.8) and $u \cdot \sigma(\mathbf{t}, \mathbf{v}; \cdot)$ is of constant sign in $[a, b]$. Since $\text{sign } u(t_l + 0) > 0$,

$$\text{sign}[u(t) \sigma(\mathbf{t}, \mathbf{v}; t)] = \text{sign } \sigma(\mathbf{t}, \mathbf{v}; t_l + 0), \quad t \in [a, b] - \{t_1, \dots, t_n\},$$

and therefore

$$\text{sign } \int_a^b u(t) \sigma(\mathbf{t}, \mathbf{v}; t) dt = \sigma_l,$$

while by (2.1) with $a_{l, v_l - 1} = 0$ and by (2.8)

$$a_{l, v_l - 2} = \int_a^b u(t) \sigma(\mathbf{t}, \mathbf{v}; t) dt.$$

This proves (2.5). The sign patterns (2.6) and (2.7) are obtained similarly. ■

The main tool in the proof of Theorem 1 is a topological argument (see, e.g., [1]).

BORSUK ANTIPODALITY THEOREM. *Let Ω be a bounded, open symmetric neighborhood of 0 in R^{m+1} and let $T \in C(\partial\Omega, R^m)$ be odd on $\partial\Omega$, i.e., $T(-x) = -T(x)$. Then there exists an $x \in \partial\Omega$ for which $T(x) = 0$.*

Proof of Theorem 1. Let $S^k \subset R^{k+1}$ be defined by

$$S^k = \left\{ \mathbf{y} = (y_0, y_1, \dots, y_k) : \sum_{i=0}^k |y_i| = b - a \right\}.$$

Given $\mathbf{y} \in S^k$ we associate with \mathbf{y} the vector $\mathbf{t} = \mathbf{t}(\mathbf{y})$ specified by $t_0 = a$, $t_i = t_{i-1} + |y_{i-1}|$, $i = 1, 2, \dots, k$ and t_i , $i = k + 1, \dots, n + 1$.

Since by this construction the nodes $t_i(\mathbf{y})$, $1 \leq i \leq k$, are not necessarily distinct, divided differences are used to extend the definition of the GQF (2.1) continuously to all possible $\mathbf{t}(\mathbf{y})$. Thus we define $\{\hat{t}_0, \hat{t}_1, \dots, \hat{t}_N\}$ to contain ν_1 times the node t_i for $i = 0, \dots, n + 1$, and enumerate the \hat{t}_j 's such that

$$\hat{t}_{N+1-i} = t_i, \quad i = 1, 2, \dots, k. \tag{2.9}$$

For $U \subset C^N[a, b]$, the divided differences $u[\hat{t}_0, \hat{t}_1, \dots, \hat{t}_j]$, $u \in U$, are continuous functions of the j arguments, in $[a, b]^j$, for $0 \leq j \leq N$ [10, p. 252].

We consider the linear functionals $l_{\mathbf{y}}$ defined by

$$l_{\mathbf{y}}(u) = \sum_{i=0}^k (\text{sign } y_i) \int_{t_i}^{t_i + |y_i|} u(t) \sigma(\mathbf{t}, \mathbf{v}; t) dt - \sum_{j=0}^N b_j u[\hat{t}_0, \hat{t}_1, \dots, \hat{t}_j]. \tag{2.10}$$

Here $\mathbf{t} = \mathbf{t}(\mathbf{y})$ and $b_j = b_j(\mathbf{y})$, $j = 0, 1, \dots, N$, are to be chosen so that

$$l_{\mathbf{y}}(u_i) = 0, \quad i = 0, 1, \dots, N. \tag{2.11}$$

Conditions (2.11) provide a linear system of equations for the $N + 1$ coefficients b_0, b_1, \dots, b_N . The matrix is nonsingular, being the adjoint of the matrix for the Hermite interpolation problem at \mathbf{t} , by functions from U . Since the matrix and the right-hand side of the equations are continuous functions of \mathbf{y} , the mapping $T: S^k \rightarrow R^k$,

$$T(\mathbf{y}) = \{b_N(\mathbf{y}), b_{N-1}(\mathbf{y}), \dots, b_{N-k+1}(\mathbf{y})\},$$

is continuous and odd. By the Borsuk Antipodal Theorem we have $T\mathbf{y}^* = T(-\mathbf{y}^*) = 0$ for a $\mathbf{y}^* \in S^k$. Formula (2.10), in view of (2.11) and the choice of \mathbf{y}^* , becomes

$$\begin{aligned} & \sum_{i=0}^k \text{sign } y_i^* \int_{t_i(\mathbf{y}^*)}^{t_i(\mathbf{y}^*) + |y_i^*|} u(t) \sigma(\mathbf{t}^*(\mathbf{y}), \mathbf{v}; t) dt \\ & = \sum_{j=0}^{N-k} b_j(\mathbf{y}^*) u[\hat{t}_1(\mathbf{y}^*), \dots, \hat{t}_j(\mathbf{y}^*)], \quad u \in U. \end{aligned} \tag{2.12}$$

Let $a < \tau_1 < \dots < \tau_s < b$, $s \leq k$, be the distinct nodes among $t_1(\mathbf{y}^*), \dots, t_k(\mathbf{y}^*)$, contained in (a, b) , and denote by $0 = \tau_0 < \tau_{s+1} < \dots < \tau_m < \tau_{m+1} = b$ those nodes among $t_0, t_{k+1}, \dots, t_n, t_{n+1}$ which do not coincide

with one of the nodes τ_1, \dots, τ_s . Then (2.12) can be written in the form (2.3) with the nodes τ , the multiplicities

$$w_i = \sum_{j \in I(i)} v_j, \quad I(i) = \{j \mid t_j(\mathbf{y}^*) = \tau_i\}, \quad i = 0, \dots, m + 1,$$

and, in view of the ordering (2.9), with

$$a_{j, w_j - 1} = 0, \quad j = 1, \dots, s.$$

Hence by Lemma 1 all the non-zero components of \mathbf{y}^* are of the same sign, and (2.12) is equivalent to a GQF of the form (2.1) satisfying (2.2) with nodes τ and multiplicities \mathbf{w} .

To conclude the proof of the theorem it remains to verify that $y_i^* \neq 0$, $i = 0, \dots, k$. Suppose to the contrary that $y_i^* = 0$ for at least one i , $0 \leq i \leq k$. Then either there exists l , $1 \leq l \leq s$, such that

$$a_{l, w_l - 1} = a_{l, w_l - 2} = 0$$

or $y_0^* y_k^* = 0$ and therefore at least one of the coefficients $a_{0, w_0 - 1}$, $a_{m+1, w_{m+1} - 1}$ vanishes. In all three cases we obtain a contradiction to Lemma 2. Therefore $\varepsilon y_i^* > 0$, $i = 0, \dots, k$ with $\varepsilon = +1$ or -1 , and the formula (2.12) is of the desired form. ■

As is obvious from the proof of Theorem 1, explicit relations between formula (2.1) and its equivalent formula in terms of divided differences are important in the study of GQFs with Gaussian nodes.

LEMMA 3. Given $a = t_0 < \dots < t_{n+1} = b$, let $\{\hat{t}_0, \dots, \hat{t}_N\}$ consist of each t_i repeated v_i times, $i = 0, \dots, n + 1$. Then for $u \in C^{|\nu| - 1}[a, b]$ with $|\nu| = \max\{v_i \mid 0 \leq i \leq n + 1\}$,

$$u[\hat{t}_0, \dots, \hat{t}_N] = \sum_{i=0}^{n+1} \sum_{j=0}^{v_i-1} C_{ij} u^{(j)}(t_i)$$

with

$$C_{i, v_i - 1} = \frac{1}{(v_i - 1)!} \prod_{\substack{j=0 \\ j \neq i}}^{n+1} (t_i - t_j)^{-v_j}.$$

Proof. The lemma is a direct consequence of the following formulas for divided differences [10, p. 252]:

$$f[t_0, t_1, \dots, t_{n+1}] = \sum_{i=0}^{n+1} f(t_i) \prod_{\substack{j=0 \\ j \neq i}}^{n+1} (t_i - t_j)^{-1},$$

$$u[\underbrace{t_0, \dots, t_0}_{v_0}, \dots, \underbrace{t_{n+1}, \dots, t_{n+1}}_{v_{n+1}}] = \left(\prod_{i=0}^{n+1} \frac{1}{(v_i - 1)!} \frac{\partial^{v_i - 1}}{\partial t_i^{v_i - 1}} \right) u[t_0, t_1, \dots, t_{n+1}]. \quad \blacksquare$$

Now we are in the position to prove a uniqueness result.

THEOREM 2. *Let the conditions of the existence and uniqueness theorem prevail and let $0 \leq k \leq n$. Given $n - k$ nodes $a < t_{k+1} < t_{k+2} < \dots < t_n < b$ there is at most one set of k nodes in (a, t_{k+1}) ,*

$$a < t_1 < t_2 < \dots < t_k < t_{k+1}, \tag{2.13}$$

for which the GQF (2.1) with the nodes t_0, t_1, \dots, t_{n+1} satisfies (2.2)

Proof. There is nothing to prove for $k = 0$. Let $k > 0$ and assume that the theorem is true for $k - 1$.

Given t_{k+1}, \dots, t_n suppose that there are two sets of k Gaussian nodes in (a, t_{k+1}) : $\{t_1^*, t_2^*, \dots, t_k^*\}$ and $\{t_1^{**}, t_2^{**}, \dots, t_k^{**}\}$. In view of the induction hypothesis $t_k^* \neq t_k^{**}$, and we may restrict ourselves to the case $t_k^* < t_k^{**}$. Define a subset Y of $[t_k^*, t_k^{**}]$ as follows: If $t_k \in Y$ then there are $k - 1$ nodes $t_1 < \dots < t_{k-1}$ in the interval (a, t_k) such that

$$a_{i, v_i - 1} = 0, \quad i = 1, 2, \dots, k - 1, \tag{2.14}$$

in (2.1) with the nodes t_0, t_1, \dots, t_{n+1} . Obviously $t_k^*, t_k^{**} \in Y$.

First we verify that Y is open in $[t_k^*, t_k^{**}]$, and then show that $Y = [t_k^*, t_k^{**}]$. To do so, we regard the $N - k + 2$ weights $\mathbf{a} = \{a_{ij}\}$ in (2.1), not prescribed by (2.14), and the $k - 1$ Gaussian nodes, t_1, \dots, t_{k-1} , as functions of t_k , to be determined implicitly by the system of equations

$$F_j(\mathbf{a}, t_1, \dots, t_{k-1}; t_k) = 0, \quad j = 0, 1, \dots, N, \tag{2.15}$$

which is obtained by applying (2.1) with (2.14), $\mathbf{a}(t_k)$, and the nodes

$$t_0, t_1(t_k), \dots, t_{k-1}(t_k), t_k, t_{k+1}, \dots, t_{n+1}$$

to $u_j, j = 0, \dots, N$.

Differentiation of (2.15) with respect to t_k , yields.

$$\begin{aligned} & \sum_{i=1}^k (\sigma_{i-1} - \sigma_i) u(t_i) \sigma(t_i) \frac{dt_i}{dt_k} \\ & - \sum_{i=0}^{n+1} \sum_{l=0}^{\hat{v}_i - 1} \left[a_{il} u^{(l+1)}(t_i) \frac{dt_i}{dt_k} + \frac{da_{il}}{dt_k} u^{(l)}(t_i) \right] = 0, \\ & u = u_j, j = 0, \dots, N, \end{aligned} \tag{2.16}$$

with

$$\begin{aligned} \hat{v}_i &= v_1 - 1, & 1 \leq i \leq k - 1, \\ &= v_i, & i = 0, \quad k \leq i \leq n + 1, \end{aligned}$$

and

$$dt_i/dt_k = 0, \quad i = 0, k + 1 \leq i \leq n + 1. \tag{2.17}$$

The Jacobian of $\mathbf{F} = \{F_j\}_{j=0}^N$ is the determinant of the linear system (2.16) for the $N + 1$ derivatives

$$dt_i/dt_k, \quad 1 \leq i \leq k - 1, \tag{2.18}$$

$$da_{ij}/dt_k, \quad j = 0, \dots, \hat{v}_i - 1, i = 0, \dots, n + 1. \tag{2.19}$$

Thus the Jacobian of \mathbf{F} is non-singular if the derivatives (2.18) and (2.19) are uniquely determined by (2.16). Indeed for $v_{ij} \in U$ satisfying the N interpolation conditions:

$$v_{ij}^{(\mu)}(t_l) = \delta_{il} \delta_{j\mu}, \quad \mu = 0, \dots, v_l - 1, l = 0, \dots, n + 1,$$

we obtain from (2.16) and (2.17) the simplified system of equations.

$$a_{i,j-1} \frac{dt_i}{dt_k} + \frac{da_{ij}}{dt_k} = -a_{k,v_k-1} v_{ij}^{(v_k)}(t_k),$$

$$j = 0, \dots, v_i - 2, i = 1, \dots, k - 1, \tag{2.20}$$

$$a_{i,v_i-2} \frac{dt_i}{dt_k} = -a_{k,v_k-1} v_{i,v_i-1}^{(v_k)}(t_k),$$

$$i = 1, \dots, k - 1, \tag{2.21}$$

$$\frac{da_{ij}}{dt_k} = -a_{k,v_k-1} v_{ij}^{(v_k)}(t_k) - a_{k,j-1} \delta_{ik},$$

$$j = 0, \dots, v_i - 1, i = 0, i = k, \dots, n + 1, \tag{2.22}$$

with the convention

$$a_{i,-1} := (\sigma_i - \sigma_{i-1}) \sigma(t_i), \quad i = 1, \dots, k - 1. \tag{2.23}$$

Equations (2.21) determine the derivatives (2.18), since $a_{i,v_i-2} \neq 0$, $i = 1, \dots, k - 1$, by Lemma 2 for $v_i \geq 2$ and by (2.23) for $v_i = 1$. The derivatives (2.19) are then determined explicitly by (2.20) and (2.22).

In particular

$$\frac{da_{k,v_k-1}}{dt_k} = -a_{k,v_k-2} - a_{k,v_k-1} v_{k,v_k-1}^{(v_k)}(t_k), \tag{2.24}$$

and for $t_k = t_k^*$

$$a_{k,v_k-1}(t_k^*) = 0, \quad \frac{da_{k,v_k-1}}{dt_k}(t_k^*) = -a_{k,v_k-2}(t_k^*). \tag{2.25}$$

Invoking the Implicit Function Theorem we conclude that $\mathbf{a}(t_k)$, $t_1(t_k), \dots, t_{k-1}(t_k)$, determined implicitly by (2.15) are differentiable functions of t_k , and hence there is $\varepsilon > 0$ such that

$$[t_k^*, t_k^* + \varepsilon) \subset Y, \quad (t_k^{**} - \varepsilon, t_k^{**}] \subset Y.$$

By the continuity of $t_1(t_k) < \dots < t_{k-1}(t_k)$, and in view of Theorem 1, $Y = [t_k^*, t_k^{**}]$ if there is no $\tilde{t}_k \in (t_k^*, t_k^{**})$ such that

$$\lim_{t_k \rightarrow \tilde{t}_k} t_{k-1}(t_k) = \tilde{t}_k. \tag{2.26}$$

Suppose to the contrary that (2.26) holds, and that $\tilde{t}_k \in [t_k^*, t_k^{**}]$ is the smallest with this property, namely that $[t_k^*, \tilde{t}_k) \subset Y$. Now by the proof of Theorem 1, for each $t_k \in [t_k^*, t_k^{**}]$, the GQF with the Gaussian nodes $t_1(t_k), \dots, t_{k-1}(t_k)$ and the prescribed nodes $t_0, t_k, t_{k+1}, \dots, t_{n+1}$ can be written in the form

$$\int_a^b u(t) \sigma(\mathbf{t}(t_k), \mathbf{v}; t) dt = \sum_{j=0}^{N-k+1} b_j(t_k) u[\hat{t}_0, \dots, \hat{t}_j], \tag{2.27}$$

where $\hat{\mathbf{t}}(t_k) = \{\hat{t}_0, \dots, \hat{t}_{N-k+1}\}$ consists of each Gaussian node repeated one time less than its multiplicity, while each prescribed node is repeated according to its multiplicity. Also by the proof of Theorem 1, the coefficients $\{b_j\}$ in (2.27) are continuous functions of t_k . Let $\hat{t}_{N-k+1} = t_k$; then for all $t_k \in (t_k^*, \tilde{t}_k)$ we obtain by Lemma 3

$$a_{k, v_k-1} = \frac{b_{N-k+1}(t_k)}{(v_k-1)!} (t_k - t_0)^{-v_0} \prod_{j=1}^{k-1} (t_k - t_j(t_k))^{-v_j+1} \prod_{j=k+1}^{n+1} (t_k - t_j)^{-v_j}. \tag{2.28}$$

On the other hand, for $t_k = \tilde{t}_k$, $t_{k-1}(\tilde{t}_k) = \tilde{t}_k$ is a Gaussian node of multiplicity $\tilde{v} := v_{k-1} + v_k$, and the coefficient of $u^{(\tilde{v}-2)}(\tilde{t}_k)$ in the GQF is given by

$$\begin{aligned} \tilde{a}_{k, \tilde{v}-2} &= \frac{b_{N-k+1}(\tilde{t}_k)}{(\tilde{v}-2)!} (\tilde{t}_k - t_0)^{-v_0} \\ &\times \prod_{j=1}^{k-2} (\tilde{t}_k - t_j(\tilde{t}_k))^{-v_j+1} \prod_{j=k+1}^{n+1} (\tilde{t}_k - t_j)^{-v_j}. \end{aligned} \tag{2.29}$$

Now for $\eta > 0$ small enough, it follows from (2.25), Lemma 2, and (2.23) that $\text{sign } a_{k, v_k-1}(t_k) = -\sigma_k$ for $t_k \in (t_k^*, t_k^* + \eta)$, and hence by (2.28)

$$\text{sign } b_{N-k+1}(t_k) = -\sigma_k (-1)^{\sum_{j=k+1}^{n+1} v_j}, \quad t_k \in (t_k^*, t_k^* + \eta). \tag{2.30}$$

If $b_{N-k+1}(\hat{t}_k) = 0$ for $\hat{t}_k < \tilde{t}_k$, we take $t_k^{**} = \hat{t}_k$, and conclude that $Y = [t_k^*, t_k^{**}]$. Otherwise we can assume that $b_{N-k+1}(t_k) \neq 0$ for $t_k \in (t_k^*, \tilde{t}_k]$. That $b_{N-k+1}(\tilde{t}_k) \neq 0$ follows from (2.29) and Lemma 2. Hence $b_{N-k+1}(t_k)$ has a constant sign in $(t_k^*, \tilde{t}_k]$. But (2.29) and Lemma 2 yield

$$\text{sign } b_{N-k+1}(\tilde{t}_k) = \sigma_k (-1)^{\sum_{j=k+1}^n v_j}, \tag{2.31}$$

in contradiction to (2.30). Thus (2.26) cannot hold for $\tilde{t}_k \in (t_k^*, t_k^{**})$, and $Y = [t_k^*, t_k^{**}]$. By choosing t_k^{**} as the first zero of $a_{k,v_k-1}(t_k)$ in $(t_k^*, t_k^{**}]$, we guarantee that for each $t_k \in (t_k^*, t_k^{**})$ there is a GQF (2.1) with nodes $t_0 < t_1(t_k) < \dots < t_{k-1}(t_k) < t_k < \dots < t_{n+1}$ and coefficients satisfying (2.14), such that $a_{k,v_k-1}(t_k) \neq 0$. Hence $a_{k,v_k-1}(t_k)$ has a constant sign in (t_k^*, t_k^{**}) , which is determined by (2.25), Lemma 2, and (2.23) to be

$$\text{sign } a_{k,v_k-1}(t_k) = -\sigma_k, \quad t_k \in (t_k^*, t_k^{**}).$$

On the other hand, (2.24) evaluated at t_k^{**} , together with Lemma 2 and (2.23), determines the sign of $a_{k,v_k-1}(t_k)$ to be $+\sigma_k$ in (t_k^*, t_k^{**}) . This provides the desired contradiction, and the uniqueness is proved. ■

The existence and uniqueness theorem of Section 1 is an immediate consequence of Theorems 1 and 2.

Before concluding this section we conjecture that the uniqueness result of Theorem 2 is valid without the restriction (2.13); namely, there is a uniqueness result for the GQFs which are known to exist by Theorem 1.

3. EXTREMAL PROPERTIES OF GENERALIZED GAUSSIAN QUADRATURE FORMULAS

In this section we study the extremal properties of the nodes of the generalized Gaussian quadrature formula (GGQF),

$$\int_a^b \sigma(\mathbf{x}, \mathbf{v}; t) f(t) dt \approx \sum_{\lambda=0}^{v_0-1} a_{0\lambda} f^{(\lambda)}(a) + \sum_{k=1}^n \sum_{\lambda=0}^{v_k-2} a_{k\lambda} f^{(\lambda)}(x_k) + \sum_{\lambda=0}^{v_{n+1}-1} a_{n+1,\lambda} f^{(\lambda)}(b), \tag{3.1}$$

which is exact for an ET-system $U = \text{span}\{u_0, \dots, u_N\}$ of dimension $N + 1 = \sum_{k=0}^{n+1} v_k$.

The sign pattern of the leading coefficients in (3.1) is obtained as a direct consequence of Lemma 2:

THEOREM 3. *Let (3.1) be exact for all the functions in U . Then*

$$\begin{aligned} \text{sign } a_{i,v_i-2} &= \text{sign } \sigma(\mathbf{x}, \mathbf{v}; x_i + 0), & \text{if } v_i > 1, i = 1, \dots, n, \\ \text{sign } a_{0,\mu} &= \text{sign } \sigma(\mathbf{x}, \mathbf{v}; a + 0), & \text{if } v_0 > 0, \mu = v_0 - 1, \\ \text{sign } a_{n+1,\mu} &= \text{sign } \sigma(\mathbf{x}, \mathbf{v}; b - 0)(-1)^\mu, & \text{if } v_{n+1} > 0, \mu = v_{n+1} - 1. \end{aligned} \tag{3.2}$$

Given the vector of multiplicities \mathbf{v} ,

$$v_0 \geq 0, \quad v_{n+1} \geq 0, \quad v_i > 0, \quad i = 1, \dots, n, \tag{3.3}$$

we consider the minimization of the functional $I_{\mathbf{v}}(\mathbf{x})$,

$$I_{\mathbf{v}}(\mathbf{x}) := \int_a^b |P(\mathbf{x}, \mathbf{v}; t)| \sigma(t) dt, \quad \mathbf{x} \in \Omega_n := \{ \mathbf{x} : a = x_0 < \dots < x_{n+1} = b \}, \tag{3.4}$$

with $P(\mathbf{x}, \mathbf{v}; t) = u_{N+1} + \sum_{k=0}^N c_k u_k$ satisfying

$$P^{(\lambda)}(\mathbf{x}, \mathbf{v}; x_i) = 0, \quad \lambda = 0, \dots, v_i - 1, i = 0, \dots, n + 1. \tag{3.5}$$

Here $u_0, \dots, u_N, \rho u_{N+1}$ is an ET-system of order $N + 2$, with $\rho = +1$ or -1 , so that

$$\text{sign } P(\mathbf{x}, \mathbf{v}; t) = \text{sign } \sigma(\mathbf{x}, \mathbf{v}; t), \quad t \in (a, b) - \{x_1, \dots, x_n\}. \tag{3.6}$$

This problem is related to the generalized Gaussian quadrature formula (3.1) by the following theorem:

THEOREM 4. *If $\xi = (a = \xi_0 < \dots < \xi_{n+1} = b) \in \Omega_n$ minimizes $I_{\mathbf{v}}(\mathbf{x})$ of (3.4) over Ω_n , then ξ_0, \dots, ξ_{n+1} are the nodes of a quadrature formula of the form (3.1) related to the multiplicities \mathbf{v} , which is exact for $U = \text{span} \{u_0, \dots, u_N\}$.*

Proof. The minimization of $I_{\mathbf{v}}(\mathbf{x})$, with $P(\mathbf{x}, \mathbf{v}, t)$ constrained by (3.5), is equivalent to the minimization of the functional.

$$\begin{aligned} I_{\mathbf{v}}(\mathbf{x}, \mathbf{c}) &:= \int_a^b \left| \left(u_{N+1} + \sum_{k=0}^N c_k u_k \right) (t) \right| \sigma(t) dt \\ &\quad - \sum_{i=0}^{n+1} \sum_{j=0}^{v_i-1} \lambda_{ij} \left(u_{N+1} + \sum_{k=0}^N c_k u_k \right)^{(j)} (x_i), \end{aligned}$$

with $\lambda_{ij}, j = 0, \dots, v_i - 1, i = 0, \dots, n$, the corresponding Lagrange multipliers. Hence ξ and the optimal vector \mathbf{c}^* satisfy.

$$\frac{\partial I_v}{\partial c_k}(\xi, \mathbf{c}^*) = 0, \quad k = 0, \dots, N, \tag{3.7}$$

$$\frac{\partial I_v}{\partial x_k}(\xi, \mathbf{c}^*) = 0, \quad k = 1, \dots, n. \tag{3.8}$$

These equations are equivalent, in view of (3.6), to

$$\int_a^b u_k(t) \sigma(\mathbf{x}, \mathbf{v}; t) dt = \sum_{i=0}^{n+1} \sum_{j=0}^{v_i-1} \lambda_{ij} u_k^{(j)}(x_i), \quad k = 0, \dots, N, \tag{3.9}$$

and

$$\sum_{j=0}^{v_k-1} \lambda_{kj} P^{(j+1)}(\mathbf{x}, \mathbf{v}; x_k) = 0, \quad k = 1, \dots, n. \tag{3.10}$$

Now, since $P(\mathbf{x}, \mathbf{v}; t)$ satisfies (3.5) and its number of zeros counting multiplicities does not exceed $N + 1$,

$$P^{(v_k)}(\mathbf{x}, \mathbf{v}; x_k) \neq 0, \quad k = 1, \dots, n,$$

while by (3.5) and (3.10) $\lambda_{k, v_k-1} P^{(v_k)}(\mathbf{x}, \mathbf{v}; x_k) = 0, k = 1, \dots, n$. Hence $\lambda_{k, v_k-1} = 0, k = 1, \dots, n$, and (3.9) is of the form (3.1) and is exact for U . ■

Following the method of proof in [11], we obtain the existence of a point $\xi \in \Omega_n$, minimizing $I_v(\mathbf{x})$, from the next two lemmas:

LEMMA 4. For given non-negative multiplicities $\mathbf{v} = (v_0, \dots, v_{n+1})$, with $v_0 > 0$, let $P(\mathbf{x}, \mathbf{v}; t)$ be defined by (3.5) and (3.6). If $\xi \in \Omega_n$ minimizes

$$I_v(\mathbf{x}) = \int_a^b |P(\mathbf{x}, \mathbf{v}; t)| \sigma(t) dt, \quad \mathbf{x} \in \Omega_n,$$

then for $\hat{\mathbf{v}} = (\hat{v}_0, \hat{v}_1, v_1, \dots, v_{n+1})$ with $\hat{v}_0 + \hat{v}_1 = v_0, \hat{v}_0 \geq 0, \hat{v}_1 > 0$, there exists $\varepsilon > 0$ small enough, such that

$$\int_a^b |P(\xi, \mathbf{v}; t)| \sigma(t) dt > \int_a^b |P(\xi, \hat{\mathbf{v}}; t)| \sigma(t) dt, \tag{3.11}$$

for all $\xi = (a, a + \eta, \xi_1, \dots, \xi_n, b) \in \Omega_{n+1}$, with $0 < \eta < \varepsilon$.

Proof. By the optimality of ξ and by Theorem 3, ξ satisfies (3.9) with $\lambda_{k, v_k-1} = 0, k = 1, \dots, n$. Applying (3.9) to $P(\xi, \mathbf{v}; t) - P(\xi, \hat{\mathbf{v}}; t) \in U$, we obtain.

$$\int_a^b [P(\xi, \mathbf{v}; t) - P(\xi, \hat{\mathbf{v}}; t)] \sigma(\xi, \mathbf{v}; t) dt = - \sum_{j=\hat{v}_0}^{v_0-1} a_{0j} P^{(j)}(\xi, \hat{\mathbf{v}}; a). \tag{3.12}$$

Hence

$$\int_a^b |P(\xi, \mathbf{v}; t)| \sigma(t) dt - \int_a^b |P(\hat{\xi}, \hat{\mathbf{v}}; t)| \sigma(t) dt = - \sum_{j=\hat{v}_0}^{v_0-1} a_{0j} P^{(j)}(\hat{\xi}, \hat{\mathbf{v}}; a) + \int_a^{a+\eta} P(\hat{\xi}, \hat{\mathbf{v}}; t) [\sigma(\xi, \mathbf{v}; t) - \sigma(\hat{\xi}, \hat{\mathbf{v}}; t)]. \quad (3.13)$$

The proof of the lemma is completed by showing that for $\varepsilon > 0$ small enough the dominating term in the right-hand side of (3.13) is $a_{0, v_0-1} P^{(v_0-1)}(\hat{\xi}, \hat{\mathbf{v}}; a)$, and that this term is negative.

Since $\{u_0, \dots, u_N\}$ and $\{u_0, \dots, u_{N+1}\}$ are ET-systems on $[a, b]$

$$\lim_{\eta \rightarrow 0} P^{(j)}(\hat{\xi}, \hat{\mathbf{v}}; t) = P^{(j)}(\xi, \mathbf{v}; t),$$

uniformly in $[a, b]$ for $j = 0, 1, \dots, v_0$. In particular for $\varepsilon > 0$ small enough

$$P^{(v_0)}(\hat{\xi}, \hat{\mathbf{v}}; t) = P^{(v_0)}(\xi, \mathbf{v}; t) + O(\eta) \neq 0, \quad a \leq t \leq a + \eta \leq a + \varepsilon, \quad (3.14)$$

since $P^{(v_0)}(\xi, \mathbf{v}; a) \neq 0$.

Now by Rolle's theorem, $P^{(v_0-j)}(\xi, \mathbf{v}; t)$ has exactly j simple zeros in $(a, a + \eta)$, $j = 1, \dots, v_0 - \hat{v}_0$, and

$$\tau_1 > \tau_2 > \dots > \tau_{v_0 - \hat{v}_0} > a, \quad (3.15)$$

where τ_j is the smallest zero of $P^{(v_0-j)}(\hat{\xi}, \hat{\mathbf{v}}; t)$ in $(a, a + \eta)$.

It follows from (3.14) and (3.15) that

$$\begin{aligned} |P^{(v_0-1)}(\hat{\xi}, \hat{\mathbf{v}}; t)| &\leq |P^{(v_0-1)}(\hat{\xi}, \hat{\mathbf{v}}; a)|, \\ \text{sign } P^{(v_0-1)}(\hat{\xi}, \hat{\mathbf{v}}; t) &= -\text{sign } P^{(v_0)}(\xi, \mathbf{v}; a), \end{aligned} \quad a \leq t < \tau_1 \quad (3.16)$$

and since for $j = 2, \dots, v_0 - \hat{v}_0$

$$P^{(v_0-j)}(\hat{\xi}, \hat{\mathbf{v}}; t) = P^{(v_0-j+1)}(\hat{\xi}, \hat{\mathbf{v}}; \hat{\tau}_j(t))(t - \tau_j), \quad a < t < \tau_j < \tau_1, \quad (3.17)$$

where $t < \hat{\tau}_j(t) < \tau_j$, we finally obtain that

$$|P^{(v_0-j)}(\hat{\xi}, \hat{\mathbf{v}}; t)| \leq |P^{(v_0-1)}(\hat{\xi}, \hat{\mathbf{v}}; a)| (\tau_1 - a)^{j-1}, \quad a \leq t \leq \tau_j, j = 1, \dots, v_0 - \hat{v}_0. \quad (3.18)$$

The bounds in (3.18) guarantee that $a_{0, v_0-1} P^{(v_0-1)}(\hat{\xi}, \hat{\mathbf{v}}; a)$ is indeed the dominating term in the right-hand side of (3.13). That this term is negative follows from (3.2), (3.16), and the observation that $\text{sign } P^{(v_0)}(\xi, \mathbf{v}; a) = \text{sign } \sigma(\xi, \mathbf{v}; a + 0)$. ■

Remark 1. The same result as Lemma 4 holds for $\hat{\mathbf{v}} = (v_0, \dots, v_n, \hat{v}_n, \hat{v}_{n+1})$, $\xi = (a, \xi_1, \dots, \xi_n, b - \eta, b) \in \Omega_{n+1}$, with $\hat{v}_n + \hat{v}_{n+1} = v_{n+1}$ and $\hat{v}_n > 0$.

LEMMA 5. *Let \mathbf{v} be non-negative multiplicities with $v_i \geq 2$ for some i , $1 \leq i \leq n$, and let $P(\mathbf{x}, \mathbf{v}; t)$ be defined by (3.5) and (3.6). If $\xi \in \Omega_n$ minimizes*

$$I_{\mathbf{v}}(\mathbf{x}) = \int_a^b |P(\mathbf{x}, \mathbf{v}; t)| \sigma(t) dt, \quad \mathbf{x} \in \Omega_n,$$

then for $\hat{\mathbf{v}} = (v_0, v_1, \dots, v_{i-1}, \hat{v}_i, \hat{v}_{i+1}, v_{i+1}, \dots, v_{n+1})$ with $\hat{v}_i > 0$, $\hat{v}_{i+1} > 0$ satisfying $\hat{v}_i + \hat{v}_{i+1} = v_i$, there exists $\varepsilon > 0$ small enough such that

$$\Delta \equiv \int_a^b |P(\xi, \mathbf{v}; t)| \sigma(t) dt - \int_a^b |P(\xi, \hat{\mathbf{v}}; t)| \sigma(t) dt > 0 \quad (3.19)$$

for all $\hat{\xi} = (a, \xi_1, \dots, \xi_{i-1}, \xi_i - \delta, \xi_i + \eta, \xi_{i+1}, \dots, \xi_n, b) \in \Omega_{n+1}$, with $\delta = \delta(\eta) > 0$ and $0 < \eta \leq \varepsilon$.

Proof. As in the proof of the previous lemma, we use the generalized quadrature formula (3.1) at the optimal nodes ξ , guaranteed by Theorem 4, to obtain

$$\Delta = - \sum_{j=0}^{v_i-2} a_{ij} P^{(j)}(\hat{\xi}, \hat{\mathbf{v}}; \xi_i) + \int_{\xi_i-\delta}^{\xi_i+\eta} P(\hat{\xi}, \hat{\mathbf{v}}; t) [\sigma(\xi, \mathbf{v}; t) - \sigma(\hat{\xi}, \hat{\mathbf{v}}; t)] dt. \quad (3.20)$$

Since $v_i \geq 2$, the sum in (3.20) is not empty, and by Lemma 2

$$\text{sign } a_{i, v_i-2} = \text{sign } P(\xi, \mathbf{v}; \xi_i + 0) = \text{sign } \sigma(\xi, \mathbf{v}; \xi_i + 0). \quad (3.21)$$

Moreover it can be shown, as in the proof of Lemma 4, that $P^{(v_i-j)}(\hat{\xi}, \hat{\mathbf{v}}; t)$ has exactly j zeros, counting multiplicities, in $[\xi_i - \delta, \xi_i + \eta]$, $0 \leq j \leq v_i$, for $\delta > 0$, $\eta > 0$ small enough, since

$$P^{(v_i)}(\hat{\xi}, \hat{\mathbf{v}}; t) = P^{(v_i)}(\xi, \mathbf{v}; \xi_i) + O(\delta + \eta) \neq 0, \quad \xi_i - \delta \leq t \leq \xi_i + \eta. \quad (3.22)$$

In particular $P^{(v_i-2)}(\hat{\xi}, \hat{\mathbf{v}}; t)$ has two simple zeros in $[\xi_i - \delta, \xi_i + \eta]$, and $P^{(v_i-1)}(\hat{\xi}, \hat{\mathbf{v}}; t)$ has a unique simple zero $\tau_1 \in (\xi_i - \delta, \xi_i + \eta)$, with all the Rolle zeros depending continuously on η, δ . Since for $\eta = 0$, $\tau_1 \in (\xi_i - \delta, \xi_i)$, while for $\delta = 0$, $\tau_1 \in (\xi_i, \xi_i + \eta)$, for each $0 < \eta < \varepsilon$ with $\varepsilon > 0$ small enough, there exists $\delta(\eta) > 0$ such that $\tau_1 = \xi_i$. For this choice of $\delta(\eta)$, the two simple zeros of $P^{(v_i-2)}(\hat{\xi}, \hat{\mathbf{v}}; t)$ in $[\xi_i - \delta(\eta), \xi_i + \eta]$, denoted by $\tau_2^L < \tau_2^R$, are separated by ξ_i . Therefore

$$\text{sign } P^{(v_i-2)}(\hat{\xi}, \hat{\mathbf{v}}; \xi_i) = - \text{sign } P^{(v_i)}(\xi, \mathbf{v}; \xi_i) = - \text{sign } \sigma(\xi, \mathbf{v}; \xi_i + 0), \quad (3.23)$$

and

$$|P^{(v_i-2)}(\hat{\xi}, \hat{\mathbf{v}}; t)| \leq |P^{(v_i-2)}(\hat{\xi}, \hat{\mathbf{v}}; \xi_i)|, \quad \tau_2^L \leq t \leq \tau_2^R. \quad (3.24)$$

In case $v_i = 2$ it is obvious that $a_{i0}P(\hat{\xi}, \hat{v}; \xi_i)$ is the dominating term in the right-hand side of (3.20). The term $a_{i, v_i - 2}P^{(v_i - 2)}(\hat{\xi}, \hat{v}; \xi_i)$ is negative for all $v_i \geq 2$, by (3.21) and (3.23). Thus to conclude the proof it is sufficient to show that this term is dominating also in case $v_i \geq 3$.

Let $\tau_j^L < \xi_i < \tau_j^R$ be the smallest and largest zero of $P^{(v_i - j)}(\hat{\xi}, \hat{v}; t)$ in $[\xi_i - \delta(\eta), \xi_i + \eta]$, $j = 2, \dots, v_i$, and let $J_j := [\tau_j^L, \tau_j^R]$. Then

$$\max_{t \in J_j} |P^{(v_i - j)}(\hat{\xi}, \hat{v}; t)| = \max_{t \in J_{j-1}} |P^{(v_i - j)}(\hat{\xi}, \hat{v}; t)|, \quad 3 \leq j \leq v_i. \quad (3.25)$$

Since each zero $\tau_{j-1}^L, \tau_{j-1}^R$ is either a Rolle zero of $P^{(v_i - j + 1)}(\hat{\xi}, \hat{v}; t)$ or an endpoint of $[\xi_i - \delta(\eta), \xi_i + \eta]$, $P^{(v_i - j)}(\hat{\xi}, \hat{v}; t)$ vanishes at least once in $[\tau_{j-1}^L, \tau_{j-1}^R]$, and hence by (3.25) and the mean-value theorem

$$\max_{t \in J_j} |P^{(v_i - j)}(\hat{\xi}, \hat{v}; t)| \leq \max_{t \in J_{j-1}} |P^{(v_i - j + 1)}(\hat{\xi}, \hat{v}; t)|(\tau_{j-1}^R - \tau_{j-1}^L), \quad 3 \leq j \leq v_i.$$

These bounds together with (3.24), yield

$$|P^{(v_i - j)}(\hat{\xi}, \hat{v}; \xi_i)| \leq |P^{(v_i - 2)}(\hat{\xi}, \hat{v}; \xi_i)|(\eta + \delta)^{j - 2}, \quad 3 \leq j \leq v_i,$$

and the proof of the lemma is completed. ■

As a direct consequence of Lemmas 4 and 5 and Remark 1, we obtain:

THEOREM 5. *Given multiplicities $\mathbf{v} = (v_0, \dots, v_{n+1})$ with $v_i > 0, i = 1, \dots, n$, and $v_0 \geq 0, v_{n+1} \geq 0$, there exists a unique point $\xi \in \Omega_n$ minimizing*

$$I_{\mathbf{v}}(\mathbf{x}) = \int_a^b |P(\mathbf{x}, \mathbf{v}; t)| \sigma(t) dt, \quad \mathbf{x} \in \Omega_n,$$

with $P(\mathbf{x}, \mathbf{v}; \cdot) = u_{N+1} + \sum_{i=0}^N c_i u_i$, satisfying (3.5) and (3.6). Moreover ξ is independent of the particular choice of u_{N+1} from the set of functions

$K(U) = \{v | \{u_0, \dots, u_N, \rho v\}$ is an ET-system on $[a, b]$, with $\rho = +1$ or $-1\}$.

Proof. $I_{\mathbf{v}}(\mathbf{x})$ is a continuous function of \mathbf{x} in Ω_n , and can be extended to $\bar{\Omega}_n$ continuously by defining $P(\mathbf{x}, \mathbf{v}; t)$ for $\mathbf{x} \in \partial\Omega_n$ of the form.

$$a = x_0 = \dots = x_{i_1 - 1} < x_{i_1} = \dots = x_{i_2 - 1} < \dots < x_{i_m + 1} = \dots = x_{n+1} = b, \quad (3.26)$$

according to the law $P(\mathbf{x}, \mathbf{v}; t) := P(\mathbf{y}, \boldsymbol{\mu}; t)$ with

$$\mu_j = \sum_{k=i_j}^{i_{j+1} - 1} v_k, \quad j = 0, \dots, m + 1 \quad (i_0 = 0, i_{m+2} = n + 2), \quad (3.27)$$

$$\mathbf{y} = (a = x_0 < x_{i_1} < \dots < x_{i_m + 1} = b) \in \Omega_m. \quad (3.28)$$

We denote the relation of type (3.27) between μ and ν by $\mu < \nu$. With this definition of $P(x, \nu; t)$ in $\bar{\Omega}_n$, $\min I_\nu(x)$ over $x \in \bar{\Omega}_n$ is attained. If the minimum is attained for $x \in \partial\Omega_n$, then it is also the minimum of $I_\mu(\cdot)$ over Ω_m , where m and $\mu < \nu$ are defined by (3.26) and (3.27). Yet, by Lemmas 4 and 5 and Remark 1, if $y \in \Omega_m$ is a solution of this minimization problem, there exists $\hat{\mu} = (\mu_0, \dots, \mu_{m+2})$, $\mu < \hat{\mu} < \nu$, and $\hat{y} \in \Omega_{m+1}$, such that $I_\nu(x) = I_\mu(y) > I_{\hat{\mu}}(\hat{y})$. Since for any $\hat{x} \in \bar{\Omega}_n$, related to \hat{y} and $\hat{\mu}$ according to (3.26)–(3.28), $I_{\hat{\mu}}(\hat{y}) = I_\nu(\hat{x})$, we arrive at a contradiction to the assumption that $I_\nu(x)$ is minimal for $x \in \partial\Omega_n$.

Therefore the minimum is attained only in Ω_n , and in view of Theorem 4 and the uniqueness of the corresponding generalized Gaussian quadrature formula with multiplicities ν , the minimum is attained at a unique point $\xi \in \Omega_n$, which is independent of the choice of u_{N+1} from $K(U)$. The proof is completed. ■

The following result is a direct consequence of the above proof:

COROLLARY 1. *Let $\nu = (\nu_0, \dots, \nu_{n+1})$ be as in Theorem 5, and let $\mu = (\mu_0, \dots, \mu_{m+1})$ satisfy $\mu < \nu$. Then*

$$\min_{x \in \Omega_n} \int_a^b |P(x, \nu; t)| \sigma(t) dt < \min_{y \in \Omega_m} \int_a^b |P(y, \mu; t)| \sigma(t) dt. \quad (3.29)$$

REFERENCES

1. H. AMANN "Lectures on Some Fixed Point Theorems," Instituto de Matemática, Rio de Janeiro, 1976.
2. D. L. BARROW, On multiple node Gaussian quadrature formulae, *Math. Comp.* **32** (1978), 431–439.
3. B. D. BOJANOV, Oscillating polynomials of least L_1 -norm, in "Numerical Integration" (G. Hammerlin, Ed.), pp. 25–33, ISNM 57, Birkhäuser, Basel, 1982.
4. B. D. BOJANOV, L_1 -approximation by Hermite interpolating polynomials, *C. R. Acad. Bulgare Sci.* **35** (1982), 1049–1053.
5. D. BRAESS AND N. DYN, On the uniqueness of monosplines and perfect splines of least, L_1 - and L_2 -norm, *J. Analyse Math.* **41** (1982), 217–233.
6. N. DYN, Generalized monosplines and optimal approximation, *Constructive Approximation* **1** (1985), 137–154.
7. C. R. HOBBY AND R. RICE, A moment problem in L_1 -approximation, *Proc. Amer. Math. Soc.* **16** (1965), 665–670.
8. A. GHIZZETTI AND A. OSSICINI, Sull'esistenza e unicità delle formule di quadrature Gaussianhe, *Rend. Mat.* **8** (1975), 1–15.
9. S. GOLUB AND J. KAUTSKY, Calculation of Gauss quadratures with multiple free and fixed knots, *Numer. Math.* **41** (1983), 147–163.
10. E. ISAACSON AND H. B. KELLER, "Analysis of Numerical Methods," Wiley, New York, 1966.

11. K. JETTER, A new type of Gaussian quadrature formulas based on Birkhoff type data, *SIAM J. Numer. Anal.* **19** (1982), 1081–1089.
12. S. KARLIN AND A. PINKUS, Gaussian quadrature formulae with multiple nodes, in “Studies in Spline Functions and Approximation Theory,” pp.113–142, Academic Press, New York, 1976.
13. S. KARLIN AND A. PINKUS, An extremal property of multiple Gaussian nodes, in “Studies in Spline Functions and Approximation Theory,” 143–162, Academic Press, New York, 1976.
14. M. KREIN, The ideas of P. L. TCHEBYSHEFF AND A. A. MARKOV in the theory of limiting values of integrals and their further developments, *Uspekhi Fiz. Nauk* (1951), 3–120 [Russian]; *Amer. Math. Soc. Transl. Ser. 2*, **12** (1951), 1–122.
15. A. PINKUS, A simple proof of the Hobby–Rice theorem, *Proc. Amer. Math. Soc.* **60** (1976), 82–84.
16. T. POPOVICIU, Aspura unei generalizari a formulei de integrare numerica a lui Gauss, *Acad. Republ. Popul. Romine Studii Cerc. Mat.* **6** (1955), 29–57.
17. L. TSCHAKALOFF, General quadrature formulae of Gaussian type, *Bulgar. Akad. Nauk. Izv. Mat. Inst.* **2** (1954), 67–84.
18. P. TURAN, On the theory of mechanical quadrature, *Acta Sci. Math. (Szeged)* **12** (1950), 30–37.